$\frac{80}{100}$

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Differential Equations

Student # [redacted] **First Exam** March, 28^{th}, 2011.

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1. Find the solution to the following IVP and determine all possible behaviors of the solution as $t \to 0$ from right. If this behavior depends on the value of y_0 give this dependence. (14 pt's.)

$$ty' + 2y = 4t^2, \qquad y(1) = y_0 .$$

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$\Rightarrow y' + \frac{2}{t} y = 4t$

$\mu(t) = e^{\int \frac{2}{t}} = e^{2\ln t} = e^{\ln t^2} = \boxed{t^2}$

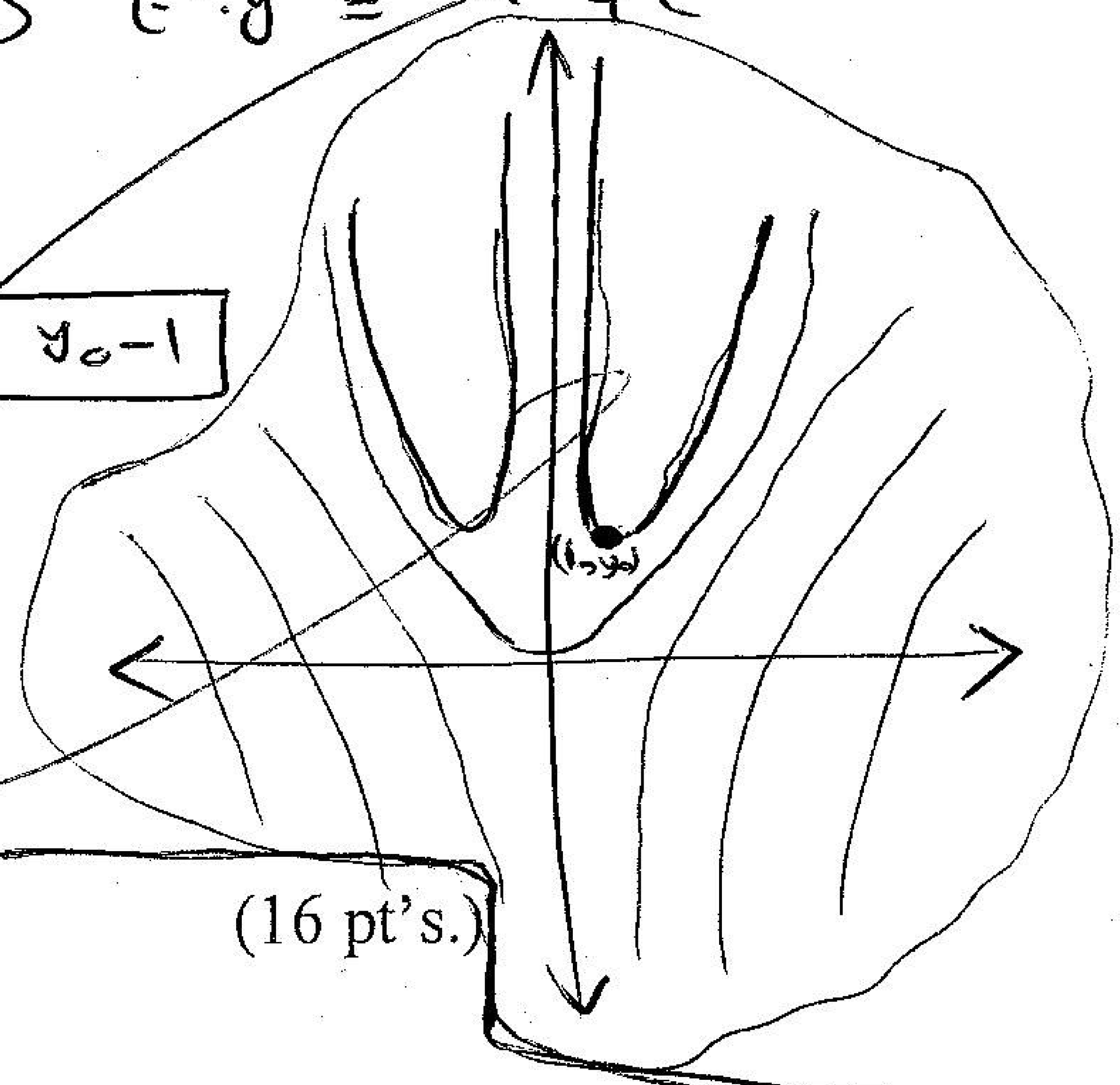
$\therefore t^2 y' + 2t y = 4t^3 \Rightarrow \frac{d}{dt}[t^2 \cdot y] = 4t^3$

$\Rightarrow \therefore t^2 \cdot y = \int 4t^3 \Rightarrow t^2 \cdot y = \frac{4t^4}{4} + c \Rightarrow t^2 \cdot y = \cancel{4t^4} + c$

$\Rightarrow \boxed{\therefore y = t^2 + \frac{c}{t^2}}$

$y(1) = y_0 \Rightarrow y_0 = 1^2 + \frac{c}{1^2} \Rightarrow y_0 = 1 + c = 0 \Rightarrow \boxed{c = y_0 - 1}$

$\boxed{\text{as } t \to 0 \Rightarrow y \to +\infty}$



2. Solve the following DE's. (16 pt's.)

a. $\frac{dy}{dx} + \frac{1}{x} y = xy^2$

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$y' + \frac{1}{x} y = x y^2 \quad \times \frac{1}{y^2} \Rightarrow y y^{-2} + \frac{1}{x} y^{-1} = x$

$\Rightarrow \boxed{\text{let } v = y^{-1}} \Rightarrow v' = -1 y^{-2} y'$

$\therefore -v' + \frac{1}{x} v = x \Rightarrow \cancel{v' - \frac{1}{x} v = x \Rightarrow \mu(t) =} \Rightarrow v' - \frac{1}{x} v = -x$

$\cancel{\Rightarrow \frac{dv}{dx} + \frac{1}{x} v = x \Rightarrow \frac{dv}{dx} = \frac{1}{x} v - x}$

$\mu(t) = e^{\int -\frac{1}{x}} = e^{-\ln x} = e^{\ln x^{-1}} = \boxed{\frac{1}{x}}$

$\therefore v' \cdot \frac{1}{x} + \frac{1}{x^2} v = -1 \Rightarrow \frac{d}{dx}\left[\frac{1}{x} \cdot v\right] = -1$

$\Rightarrow \left[\frac{1}{x} \cdot v\right] = \int -1\, dx \Rightarrow \frac{v}{x} = -x + c \Rightarrow \therefore v = -x^2 + cx$

but $v = y^{-1}$ $\therefore$ $\boxed{y = \frac{1}{-x^2 + cx}}$

$v = \frac{y}{x}$ $\quad y = xv + v$

$y' = \frac{3v^2 - 1}{2v}$ $\quad y = vx \Rightarrow y' = v + v\frac{dv}{dx}$

$$\overbrace{-(2xy)}^{M} + \overbrace{(3y^2 - x^2)}^{N} \frac{dy}{dx} = 0$$

b. $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$. (13 pt's.) 2

$2xy \frac{dy}{dx} = 3y^2 - x^2$ $\Rightarrow$ Exact D.E. we can solve it like them

$M = 3y^2 - x^2 \Rightarrow M_y = 6y$
$N = 2xy \Rightarrow N_x = 2y$ $\Big] \rightarrow$ Not exact

$\mu(t) = e^{\int \frac{M_y - N_x}{N}} = \frac{6y - 2y}{2xy} = e^{\int \frac{4y}{2xy}} = \frac{2}{x} \Rightarrow \boxed{\mu(x) = x^2}$

and Solve

c. $xydx + (2x^2 + 3y^2 - 20)dy = 0$. (17 pt's.) 11

$\underset{M}{(xy)} + \underset{N}{(2x^2 + 3y^2 - 20)} \frac{dy}{dx} = 0$

$M_y = x$
$N_x = 4x$ $\Big] \rightarrow$ Not exact

Integrating factor $\mu(y) = \frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \boxed{\frac{3}{y}}$

$\therefore \frac{d\mu}{dy} = \frac{N_x - M_y}{M} \cdot \mu \Rightarrow \frac{d\mu}{\mu} = \frac{3}{y} dy$

$\Rightarrow \mu(y) = e^{\int \frac{3}{y}} = e^{3\ln y} = e^{\ln y^3} = \boxed{y^3}$

$\underset{M}{(xy^4)} + \underset{N}{(2x^2y^3 + 3y^5 - 20y^3)} \frac{dy}{dx} = 0 \Rightarrow M_y = 4xy^3$
$N_x = 4xy^3$ $\Big] \rightarrow$ exact

~~$\psi_x = (xy^4)$ $4xy^3$~~

~~$\psi(x,y) = \frac{4x^2y^3}{2} + h(y) = 2x^2y^3 + h(y)$~~

~~$\psi_y = 3x^2y^2 + h'(y) = (2x^2y^3 + 3y^5 - 20y^3)$~~

~~$\psi_y = 6x^2y^2$~~ $+ h'(y) =$ $\rightarrow$ Exact Method | another method

$$\boxed{\psi(x,y) = \int M_x \, dx + \int n(y) dy}$$

$n(y)$: the terms of N_x that doesn't contain x

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$\therefore \psi(x,y) = \int 4xy^3 \, dx + \int (3y^5 - 20y^3) dy$

$= \frac{4x^2}{2}y^3 + \frac{3y^6}{6} - \frac{20y^4}{4} = \boxed{2x^2y^3 + \frac{y^6}{2} - 5y^4 = C}$

3. a. Solve the IVP $y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 2$. (8 pt's.)

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$r^2 - 2r + 5 = 0$

$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = \boxed{1 \pm 2i}$

$r = 1 + 2i$

$r = 1 - 2i$

$\therefore y = e^{\alpha t}[c_1 \cos \mu t + c_2 \sin \mu t] = \boxed{c_1 e^t \cos 2t + c_2 e^t \sin 2t}$

$y(\frac{\pi}{2}) = 0 \Rightarrow c_1 e^{\frac{\pi}{2}} \cos \pi + c_2 e^{\frac{\pi}{2}} \sin \pi = 0 \Rightarrow -c_1 e^{\frac{\pi}{2}} = 0$

$\Rightarrow c_1 = 0$

$y' = c_1 [2e^t(-\sin 2t) + \cos 2t\, e^t] + c_2 [2e^t \cos 2t + \sin 2t\,(e^t)]$

$y'(\frac{\pi}{2}) = c_1 [2e^{\frac{\pi}{2}}(-\sin \pi) + \cos \pi\, e^{\frac{\pi}{2}}] + c_2 [2e^{\frac{\pi}{2}} \cos \pi + \sin \pi (e^{\frac{\pi}{2}})]$

$= c_1 [0 + -e^{\frac{\pi}{2}}] + c_2 [-2e^{\frac{\pi}{2}} + 0] = 2$

$-2c_2 e^{\frac{\pi}{2}} = 2 \Rightarrow c_2 e^{\frac{\pi}{2}} = -1 \Rightarrow c_2 = \frac{-1}{e^{\frac{\pi}{2}}} = \boxed{-e^{\frac{-\pi}{2}}}$

$\boxed{\therefore y = -e^{\frac{-\pi}{2}} \cdot e^t \sin 2t}$

$t^{-2} \quad \frac{t^{-2+1}}{-2+1} \cdot 2 = \frac{2t^{-1}}{-1} = \frac{-2}{t}$

b. If y_1 and y_2 are two linearly independent solutions of $t^2 y'' - 2y' + (3+t)y = 0$ and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(4)$. (10 pt's.)

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$y'' - \frac{2}{t^2} y' + \frac{3+t}{t^2} y = 0$

$W = Ce^{-\int p(t)\,dt} = e^{-\int -\frac{2}{t^2}} = e^{\int \frac{2}{t^2}} = e^{\int 2t^{-2}} = e^{2\int t^{-2}} = \boxed{\cancel{e^{\frac{-2}{t}+c}}}$

$\Rightarrow W(y_1, y_2)(2) = e^{\frac{-2}{2}+c} = 3 \Rightarrow e^{-1+c} = 3 \Rightarrow \ln e^{-1+c} = \ln 3$

$\Rightarrow -1 + c = \ln 3 \Rightarrow \boxed{c = \ln 3 + 1}$

$\therefore W(y_1, y_2)(4) = e^{\frac{-2}{4} + \ln 3 + 1} = e^{\frac{1}{2} + \ln 3} = e^{\frac{1}{2}} \cdot e^{\ln 3} = \boxed{3e^{\frac{1}{2}}}$

c. Find the longest interval in which the IVP $t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0,\ y'(3) = -1$ is certain to have a unique solution. **Do not attempt to find the solution.** (6 pt's.)

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$y'' + \frac{3t}{t(t-4)} y' + \frac{4}{t(t-4)} y = \frac{2}{t(t-4)}$

$\Rightarrow t(t-4) = 0 \Rightarrow t = 0,\ t = 4$

$(-\infty, 0) \cup (0, 4) \cup (4, \infty)$

The longest interval is $\boxed{0 < t < 4}$

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3. If the function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution of the DE on $(0, \infty)$. (16 pt's.)

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$y = x^2 v$

$y' = x^2 v' + (2x) v$

$y'' = x^2 v'' + v'(2x) + (2x) v' + v(2) = x^2 v'' + 4x v' + 2v$

$\therefore \quad x^2(x^2 v'' + 4x v' + 2v) - 3x(x^2 v' + 2xv) + 4(x^2 v) = 0$

$(x^4 v'' + 4x^3 v' + 2v x^2) - (3x^3 v' + 6x^2 v) + 4x^2 v = 0$

$v''(x^4) + v'(4x^3 - 3x^3) + v(2x^2 + 6x^2 + 4x^2) = 0$

$v''(x^4) + v'(x^3) + \cancel{v(12x^2)} = 0$

$\therefore \quad x^4 v'' + x^3 v' + \cancel{12x^2 v} = 0$

$v'' + x^{-1} v' + \cancel{12x^{-2} v} = 0$

let $w = v' \Rightarrow w' = v''$

$w' + \frac{1}{x} w = 0$

$\mu(t) = e^{\int \frac{1}{x}} = e^{\ln x} = \boxed{x}$

$x w' + \frac{1}{x} \cdot x \, w = 0$

$\frac{d}{dx}[w \cdot x] = \int 0$

$w \cdot x = c \Rightarrow \therefore w = \frac{c}{x}$

but $w = v' \Rightarrow \therefore v' = \frac{c}{x} \Rightarrow v = \int \frac{c}{x} = c \ln(x) + d$

$\therefore \quad v = c \ln(x) + d$

but $y = x^2 v \Rightarrow \boxed{\therefore y = cx^2 \ln x + \boxed{x^2 d}}$ repeated

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